LECTURE NOTES ON YOUNG TABLEAUX AND COMBINATORICS

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ABSTRACT. This document serves as the class notes for Young Tableaux and Combinatorics class taught by Shiyue Li in Week 3 of Canada/USA Mathcamp 2019. These notes are based on Kevin Carde's class in 2017 on Combinatorics of Young Tableaux.

1. DAY 1

1.1. Standard/Semistandard Young Tableaux, RS/RSK Correspondence.

Definition 1.1. Given an positive integer $n > 0$, a **partition** λ of n is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_d)$ such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \lambda_d > 0.$$

and $\sum \lambda_i = n$. If a sequence λ is a partition of n, we denote $\lambda \vdash n$.

Definition 1.2. Given a partition $\lambda \vdash n$, a **Young diagram** λ of size n is an left-aligned arrangement of n boxes such that the i-th row contains λ_i boxes. We use "shape" and "Young diagram" interchangeably.

Example 1.3. The Young diagram corresponding to the partition $(3,2)$ of 5 is as follows.

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}$$

Proposition 1.4. *Given $n > 0$, the set of all Young diagrams of size n is in bijection with partitions of n.*

Definition 1.5. A **standard Young tableau** (SYT) of a Young diagram $\lambda \vdash n$ is a bijective map from the boxes of λ to $[n]$ such that all rows and columns are strictly increasing. In other words, it is a filling of the boxes in λ using numbers in $[n]$.

Example 1.6. There are 5 standard Young tableaux associated with the shape $(3,2)$. They are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}.$$

Definition 1.7. Given $n > 0$, a **composition** μ of n, is sequence of numbers $(\mu_1, \ldots, \mu_k)$ such that $\sum \mu_i = n$.

Definition 1.8. Given a shape $\lambda \vdash n$, a composition $\mu = (\mu_1, \ldots, \mu_k)$ and a **semistandard** Young tableau (SSYT) of λ is a filling of λ using content μ such that the followings hold:

- The number i appears μ_i times.
- All rows are weakly increasing.
- All columns are strongly increasing.

There are many combinatorial questions that we would like to answer in this class regarding/using combinatorial theory of Young tableaux.

(a) Given $n > 0$, what's the relation between f^λ and n?
(b) How many standard Young tableaux are there given a shape $\lambda \vdash n$?
(c) How many semistandard Young tableaux are there given a shape $\lambda \vdash n$ and a composition μ of n?
(d) *USAMO 2016 #2*: For any positive integer k, show that

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

Notation 1.9. Given $n > 0$ and $\lambda \vdash n$, let f^λ be the number of standard Young tableaux of shape λ.

Today we are devoted to answering the first question.

Observation 1.10. (a) When $n = 1$, we only have the SYT

$$\begin{ytableau} 1 \end{ytableau}.$$

(b) When $n = 2$, we have the SYT

$$\begin{ytableau} 1 \\ 2 \end{ytableau} \quad \text{and} \quad \begin{ytableau} 1 & 2 \end{ytableau}.$$

(c) When $n = 3$, we have the SYT

$$\begin{ytableau} 1 & 2 & 3 \end{ytableau}, \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}, \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}, \quad \begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}$$

(d) When $n = 4$, we have the SYT,

$$\begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau}, \quad \begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau}, \quad \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}, \quad \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}, \quad \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}, \quad \begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau}, \quad \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau}, \quad \begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau}, \quad \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau}, \quad \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau}.$$

Notice that in this case,

$$f^{\ydiagram{4}} = 1, f^{\ydiagram{3,1}} = 3,$$

$$f^{\ydiagram{2,2}} = 2, f^{\ydiagram{2,1,1}} = 3,$$

$$f^{\ydiagram{1,1,1,1}} = 1.$$

We might observe that

$$4! = 24 = \sum_{\lambda \vdash n} (f^\lambda)^2 = 1^2 + 3^2 + 2^2 + 3^2 + 1.$$

Proposition 1.11. *Given $n > 0$,*

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

To prove this proposition, we establish a correpondence between symmetric group elements and standard Young tableaux.

Notation 1.12. Given a symmetric group S_n and an element σ, we can write σ in **one-line notation** as follows:

$$\sigma := \sigma(1)\,\sigma(2)\cdots\sigma(n).$$

Example 1.13. The symmetric group S_3 contains elements $123, 213, 132, 312, 231, 312$.

Example 1.14. We can try using the one-line notation of S_3 to construct standard Young tableaux.

Consider $123 \in S_3$. We can put them in the Young diagram $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}$ in a row as they are to get the standard Young tableau $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$.

Consider the next element $213 \in S_3$. We can imitate the previous process to get a $\begin{array}{|c|}\hline 2 \\ \hline\end{array}$ when 2 comes in. Inserting 1, we have no choice but bump 2 to somewhere else. We can bump 2 to the right, or to the next row; we get to decide on how we do this, as long as we do it consistently and systematically. Let us bump 2 to the second row, and hence we obtain $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$. Inserting 3, we get $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$.

Constructing standard Young tableaux using the above algorithm, for all S_3 elements, we have the following map from S_3 to the set of standard Young tableaux of size 3:

$$\begin{aligned}
123 &\mapsto \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}, \\
213 &\mapsto \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \\
132 &\mapsto \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \\
321 &\mapsto \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}, \\
231 &\mapsto \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \\
312 &\mapsto \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}.
\end{aligned}$$

But the map is far from injective! The permutations 132 and 312 give the same standard Young tableaux under this algorithm. But something is different between them. If we track the steps in the construction of the SYT of 132, we have

$$132 \mapsto \begin{array}{|c|}\hline 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

If we track the steps in the construction of the SYT of 312, we have

$$312 \mapsto \begin{array}{|c|}\hline 3 \\ \hline\end{array} \to \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

Observation 1.15. We have the following observations.

(a) The permutations that result in the same SYT construct the SYT in distinctly different orders.

(b) The arrangement of the boxes that record the location of i-th box being constructed is a SYT, for $i = 1, \dots, n$. For example, recording the location of the i-th box in the SYT of 132, we have $\ytableaushort{12,3}$; we have $\ytableaushort{13,2}$ for recording locations in the SYT of 312. If we pair these "location-recording SYT" with the previous set of SYT, we get a bijection between S_3 and the pairs of SYT of the same shape with size 3.

$$
\begin{aligned}
123 &\mapsto \ytableaushort{123}, \ytableaushort{123}\\
213 &\mapsto \ytableaushort{13,2}, \ytableaushort{13,2},\\
132 &\mapsto \ytableaushort{12,3}, \ytableaushort{12,3}\\
321 &\mapsto \ytableaushort{1,2,3}, \ytableaushort{1,2,3}\\
231 &\mapsto \ytableaushort{13,2}, \ytableaushort{12,3}\\
312 &\mapsto \ytableaushort{12,3}, \ytableaushort{13,2}.
\end{aligned}
$$

Proposition 1.16 (Robinson-Schensted Algorithm)**.** *Given $n > 0$, we construct pairs of SYT for $\sigma \in S_n$. Write $\sigma = s_1 \cdots s_n$ using one-line permutation. Let us call the SYT on the left the **insertion tableau** and the SYT on the right the **recording tableaux**. Starting with $(\varnothing, \varnothing)$, insert number s_i in the one-line notation of σ for each $i = 1, \dots, n$:*

- *If s_i is bigger than all numbers in a row, append s_i at the right end of the row in the insertion tableau; append i at the right end on the same row in the recording tableau.*
- *If there exists a number that is bigger than s_i, replace the minimum of such number with s_i and insert the replaced number in the next row as above; append i at the same location as where s_i ends up.*

Since σ has finite length, this algorithm terminates and we obtain a pair of SYT of the same shape.

Theorem 1.17 (Robinson-Schensted Correspondence)**.** *Given $n > 0$, the following insertion algorithm performed on the one-line notation of S_n gives a bijection between permutations in S_n and the pairs of standard Young tableaux of the same shape with size n.*

Corollary 1.18. *By RS Correspondence,*

$$n! = |S_n| = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

1.2. **Exercises for Day 1.**

Exercise 1.19. Carry out RS algorithm for the permutation 748261539.

Exercise 1.20. The **inverse** of a permutation π is defined as $\pi^{-1}(b) = a$ whenever $\pi(a) = b$. Conjecture a relationship between the results of RS on π and the results of RS on π^{-1}.

Exercise 1.21. This exercise walks you through the **Robinson-Schensted-Knuth Correspondence**.

(a) Consider a $2 \times n$ matrix

$$\begin{bmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n \end{bmatrix},$$

where s_i and t_i are positive numbers no greater than n, t_i's are weakly increasing, s_i's are ordered. Starting with the empty tableaux $(\varnothing, \varnothing)$ and insert the pair (s_1, t_1) to the **insertion tableau** on the left and **recording tableau** on the right. What do you get?

(b) Turn the matrix above into a $\mathbb{N}$-**matrix** by doing the following: count the number of columns with values (a, b) appeared in the matrix, record the count at the ab-entry of a new matrix.

For example, try turning the matrix

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

into a $\mathbb{N}$-matrix.

(c) The algorithm in part (a) gives a bijection between $\mathbb{N}$-matrices and pairs of semistandard Young tableaux.

(d) Given an $\mathbb{N}$-matrix M, how big is the shape of the resulting SSYT under RSK?

(e) Suppose an $\mathbb{N}$-matrix M corresponds to a pair of SSYT (P, Q). In terms of M, how many 1s are in P? How many 2s, 3s, etc.? What about in Q?

2. Day 2

2.1. The Cauchy Product, Schur Functions and Counting SSYT.

Example 2.1. Consider the matrix

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Observe that the columns are in lexicographic order, prioritizing the second row.

Inserting each column into a pair of insertion tableau and recording tableau, we obtain a pair of semistandard Young tableaux.

$$\left(\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \right)$$

Definition 2.2. A **two-line array** is a $2 \times n$ matrix

$$\begin{bmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n \end{bmatrix}$$

such that

- $s_i, t_i \in [n]$, for all $i = 1, \ldots, n$;
- for any i, j, the column (s_i, t_i) appears later than (s_j, t_j) if $t_i > t_j$, or $t_i = t_j$ and $s_i \geq s_j$ (i.e. the columns are sorted lexicographically prioritizing the second row).

Theorem 2.3 (Robinson-Schensted-Knuth Correspondence)**.** *There is a bijection between the two-line arrays and the pairs of semistandard Young tableaux of the same shape.*

Observation 2.4. It suffices to record number of copies for each column, or each pair of number that represents a column.

Imagine in a world where you are only allowed to construct two-line arrays with the column $(1, 1)$. How many two line arrays can you have? You can have 1 empty two-line array, 1 two-line array of size 2×1, 1 two-line array of size 2×2; in general, 1 two-line array of size $2 \times n$ for every $n \in \mathbb{Z}$.

If we record one copy of the column $(1,1)$ by x_1y_1 and think of concatenation as multiplication of these monomials, we can have the power series in x_1y_1:

$$1 + x_1y_1 + (x_1y_1)^2 + \cdots = \frac{1}{1 - x_1y_1}.$$

The coefficient of the term $(x_1y_1)^n$ counts the number of two-line arrays that we can construct from concatenating n copies of $(1,1)$.

Example 2.5. Consider the term $x_1x_2^2y_1y_2y_3$ in the **Cauchy product**

$$\prod_{i,j} \frac{1}{1 - x_iy_j}.$$

This term has coefficient 3, corresponding to three two-line arrays correspond to $x_1x_2^2y_1y_2y_3$.

Proposition 2.6. *The coefficient of the term $\prod x_i^{p_i}y_j^{q_j}$ counts two line arrays where i appears exactly p_i times in the first row and j appears exactly q_j times in the second row.*

2.2. **Exercises for Day 2.**

Exercise 2.7. Consider the term $x_1x_2^2y_1y_2y_3$ in the Cauchy product

$$\prod_{i,j} \frac{1}{1 - x_iy_j}.$$

(a) What is the coefficient of this term in the product, by expanding the product into product of power series?
(b) Write down the two-line arrays corresponded with this monomial.
(c) Construct pairs of SSYT using the two-line arrays.

Exercise 2.8. Given a shape $\lambda \vdash n$, show that the number of standard Young tableaux is equal to the coefficient of $x_1x_2\cdots x_n$ in the Schur function

$$s_\lambda(\underline{x}) = \sum_{t \in \mathrm{SSYT}_\lambda} \underline{x}^t.$$

Exercise 2.9. We study the general properties of Schur functions.

(a) Consider the shape $\lambda = \square\square$. Write down explicitly the Schur function $s_{\lambda=\square\square}(x_1, x_2)$?
(b) Consider the shape $\lambda = \square\square\square$. Write down explicitly the Schur function $s_{\lambda=\square\square\square}(x_1, x_2, x_3)$?
(c) In genereal, given $n > 0$ and the shape

$$\lambda = (n) = \underbrace{\square\square\cdots\square\square}_{n \text{ times}},$$

how many SSYT are there, in terms of n?
(d) What properties does s_λ have for the shape any $n > 0$, and

$$\lambda = (n) = \underbrace{\square\square\cdots\square\square}_{n \text{ times}}?$$

(*Hint:* What happens when you permute the subscripts of the variables?)

We realize that Schur functions generalize **symmetric functions** to infinite power series.

3. Day 3

3.1. Hillman-Grassl Algorithm. Today we start answering the third question in our list: Given a shape $\lambda \vdash n$, how many standard Young tableaux are there?

The question is answered by the celebrated hook length formula.

Theorem 3.1. *Given a shape $\lambda \vdash n$, the number of standard Young tableaux of the shape λ is*

$$f^{\lambda} = \frac{n!}{\prod_h h}$$

where h ranges over all hook lengths of λ.

Example 3.2. Recall the example we saw on Day 1. There are 5 standard Young tableaux associated with the shape $(3,2)$. They are

$$\begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 \end{ytableau}\,, \quad \begin{ytableau} 1 & 3 & 4 \\ 2 & 5 \end{ytableau}\,, \quad \begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \end{ytableau}\,.$$

Indeed, by the Hook Length Formula:

$$f^{\ydiagram{3,2}} = \frac{5!}{4 \cdot 3 \cdot 2}.$$

Definition 3.3. Given a $\lambda \vdash n$, a **reverse plane partition** is a filling of λ with non-negative natural numbers such that all rows and columns are weakly increasing.

Algorithm 1: Hillman-Grassl Algorithm

Result: A λ-tableau filled with 0 and another λ-tableau with nonzero entries

Start with the southwest non-zero entry of the tableau A;

Create an empty λ-tableau, called B;

while *there exists nonzero entries in A* **do**

    Start with the south-west most box that is nonzero;

    while *the box above is the same or the box to the right is nonzero* **do**

        if *the box above is the same* **then**

            move up;

        else if *the box to the right is nonzero* **then**

            move to the right;

    end

    Subtract 1 from all the boxes traversed;

    Write 1 in the box at the same column as the starting column and at the same row as the ending row in B

end

For every hook length h, we define a geometric series in the variable q:

$$1 + q^h + q^{2h} + \cdots = \frac{1}{1-q^h}.$$

Each choice of hook length is independent from choices, all possible Hillman-Grassl tableaux is counted by

$$\prod_h \frac{1}{1-q^h}$$

where h ranges over all hook lengths of λ.

For each reverse plane partition, we add i to the ith row to turn it into a SSYT. The total number that we have added is

$$d(\lambda) = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \cdots.$$

All reverse plane partitions are thus counted by

$$q^{-d(\lambda)} s_\lambda(q^1, q^2, q^3, \cdots).$$

Proposition 3.4. *Given a shape $\lambda \vdash n$, we have*

$$\prod_h \frac{1}{1-q^h} = q^{-d(\lambda)} s_\lambda(q^1, q^2, q^3, \cdots).$$

3.2. **Exercises for Day 3.**

Exercise 3.5. This exercise walks you through the **dual Robinson-Schensted-Knuth Correspondence**.

Consider a $n \times n$ $(0,1)$-matrix (i.e. a matrix with entries 0 or 1). For example,

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(a) Turn the matrix into a **two-line array** by interpreting the entry a_{ij} in the matrix as having a_{ij} copies of the column $(i\ j)$ and then sorting the columns in lexicographic order.
(b) Produce a pair of tableaux using the same insertion algorithm together with the recording tableau; but instead of bumping the minimum number larger than the insertee, bump the maximum number smaller than the insertee.
(c) What can you say about the pair of tableaux that you obtain?
(d) Prove the identity:

$$\prod_{i,j}(1 + x_i y_j) = \sum_\lambda s_{\lambda^t}(\underline{x}) s_\lambda(\underline{y}),$$

where λ^t is the **transpose** of the shape λ (i.e. a shape obtained by flipping λ through the diagonal). The right hand side is called the **dual Cauchy product**.

4. Day 4

4.1. **Coefficients and Limits.** Recall that we proved the followings.

(a) Using two-line arrays, we showed that the **Cauchy identity of Schur functions**

$$\prod_{i,j} \frac{1}{1 + x_i y_j} = \sum_\lambda s_\lambda(\underline{x}) s_\lambda(\underline{y}).$$

(b) Using $(0,1)$-matrices, we showed that the **dual Cauchy identity of Schur functions** is

$$\prod_{i,j}(1+x_i y_j) = \sum_\lambda s_\lambda(\underline{x}) s_{\lambda^t}(\underline{y}).$$

We will plug $y_j = (1-q)q^j$ for each j in the dual Cauchy identity. On the Schur function side, the Schur function associated with the transposed tableau becomes

$$\begin{aligned} s_{\lambda^t}((1-q)q,(1-q)q^2,\ldots) &= (1-q)^n s_{\lambda^t}(q^1,q^2,\ldots) \\ &= \prod_h \frac{1-q}{1-q^h} \\ &= \prod_h \frac{1}{1+q^1+q^2+q^3+\cdots+q^{h-1}}. \end{aligned}$$

Taking the limit, the right hand side of the dual Cauchy identity becomes

$$\sum s_\lambda(\underline{x}) \prod_h \frac{1}{h}.$$

On the Cauchy product side, we have

$$\begin{aligned} \lim_{q\to 1}\prod_{i,j}(1+x_i(1-q)q^j) &= \lim_{N\to\infty}\prod_{i=1}^{\infty}\prod_{j=1}^{N}\left(1+\frac{x_i}{N}(1-\frac{1}{N})^j\right) \\ &= \lim_{N\to\infty}\prod_{i=1}^{\infty}\left(1+\frac{x_i}{N}\right)^N \\ &= \prod_{i=1}^{\infty}\lim_{N\to\infty}\left(1+\frac{x_i}{N}\right)^N \\ &= \prod_{i=1}^{\infty} e^{x_i} \\ &= e^{\sum_i x_i}. \end{aligned}$$

4.2. Exercises for Day 4.

Exercise 4.1. Prove that

$$\sum_{\lambda\vdash n} f^\lambda s_\lambda(\underline{x}) = \left(\sum_i x_i\right)^n.$$

(*Hint:* Take the coefficient of the term $y_1\cdots y_n$ on both sides of the Cauchy identity of Schur functions.)

5. Day 5

Today we move on to proving the Hook Length Formula.

Recall that we proved the following identity.

(a) The Cauchy identity for Schur functions:

(b) The dual Cauchy identity for Schur functions:

$$\prod_{i,j}(1+x_i y_j) = \sum_{\lambda\vdash n} s_\lambda(\underline{x}) s_\lambda(\underline{y}).$$

(c) Plugging in $y_j = (1-q)q^j$ for both side of the dual Cauchy identity, and taking the limit as $q \to 1$, we have

$$\sum_{\lambda \vdash n} \frac{f^\lambda}{n!} s_\lambda(\underline{x}) = \sum_{\lambda \vdash n} \prod_h \frac{1}{h} s_\lambda(\underline{x}).$$